## Adjust Hunziker et al. (2015) for TM/TE-split

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`Hun15 refers to Hunziker et al. (2015), Book refers to the derivation of Ziolkowski and Slob (2017).`

The modeller `empymod` returns the total field, hence not distinguishing between TM and TE mode, and even less between up- and down-going fields. The reason behind this is simple: The derivation of Hun15, on which `empymod` is based, returns the total field. Internally it also calculates TM and TE modes, and sums these up. However, the separation into TM and TE mode introduces a singularity at $\kappa =$. It has no contribution in the space-frequency domain to the total fields, but it introduces non-physical events in each mode with opposite sign (so they cancel each other out in the total field). In order to obtain the correct TM and TE contributions one has to remove these non-physical parts.

To remove the non-physical part we use the file `tmtemod.py` in this directory. This routine is basically a heavily simplified version of `empymod` with the following limitations:

- x-directed electric sources and electric receivers;
- only frequency domain;
- direct field is always calculated in the wavenumber-domain;
- the Fast Hankel transform is used with a 201 pt filter;
- source and receivers have to be in the same layer;
- the model must have more than one layer (there is only direct field contribution anyway for a fullspace); and
- electric permittivity and magnetic permeability are isotropic.

So `tmtemod.py` returns the signal separated into TM$^{++}$, TM$^{+-}$, TM$^{-+}$, TM$^{--}$, TE$^{++}$, TE$^{+-}$, TE$^{-+}$, and TE$^{--}$ as well as the direct field TM and TE contributions. The first superscript denotes the direction in which the field diffuses towards the receiver and the second superscript denotes the direction in which the field diffuses away from the source. For both the plus-sign indicates the field diffuses in the downward direction and the minus-sign indicates the field diffuses in the upward direction. The routine uses `empymod` wherever possible, see the corresponding functions in `empymod` for more explanation and documentation regarding input parameters.

We start with equation (105) in Hun15:

$$\begin{aligned}\hat{G}^{ee}_{xx}(\boldsymbol{x},\boldsymbol{x'},\omega) = \hat{G}^{ee;i}_{xx;s}(\boldsymbol{x}-\boldsymbol{x'},\omega) &+ \frac{1}{8\pi}\int_{\kappa=0}^{\infty}\left(\frac{\Gamma_s\tilde{g}^{tm}_{hh;s}}{\eta_s} - \frac{\zeta_s\tilde{g}^{te}_{zz;s}}{\bar{\Gamma}_s}\right)J_0(\kappa r)\kappa\,\mathrm{d}\kappa \\ &- \frac{\cos(2\phi)}{8\pi}\int_{\kappa=0}^{\infty}\left(\frac{\Gamma_s\tilde{g}^{tm}_{hh;s}}{\eta_s} + \frac{\zeta_s\tilde{g}^{te}_{zz;s}}{\bar{\Gamma}_s}\right)J_2(\kappa r)\kappa\,\mathrm{d}\kappa\ . \end{aligned} \tag{1}$$

Ignoring the incident field, and using $J_2 = \frac{2}{\kappa r}J_1 - J_0$ to avoid $J_2$-integrals, we get

$$\begin{aligned}\hat{G}^{ee}_{xx}(\boldsymbol{x},\boldsymbol{x'},\omega) = &\frac{1}{8\pi}\int_{\kappa=0}^{\infty}\left(\frac{\Gamma_s\tilde{g}^{tm}_{hh;s}}{\eta_s} - \frac{\zeta_s\tilde{g}^{te}_{zz;s}}{\bar{\Gamma}_s}\right)J_0(\kappa r)\kappa\,\mathrm{d}\kappa \\ &+ \frac{\cos(2\phi)}{8\pi}\int_{\kappa=0}^{\infty}\left(\frac{\Gamma_s\tilde{g}^{tm}_{hh;s}}{\eta_s} + \frac{\zeta_s\tilde{g}^{te}_{zz;s}}{\bar{\Gamma}_s}\right)J_0(\kappa r)\kappa\,\mathrm{d}\kappa \\ &- \frac{\cos(2\phi)}{4\pi r}\int_{\kappa=0}^{\infty}\left(\frac{\Gamma_s\tilde{g}^{tm}_{hh;s}}{\eta_s} + \frac{\zeta_s\tilde{g}^{te}_{zz;s}}{\bar{\Gamma}_s}\right)J_1(\kappa r)\,\mathrm{d}\kappa\ . \end{aligned} \tag{2}$$

From this the TM- and TE-parts follow as

$$\mathrm{TE} = \frac{\cos(2\phi)-1}{8\pi}\int_{\kappa=0}^{\infty}\frac{\zeta_s\tilde{g}^{te}_{zz;s}}{\bar{\Gamma}_s}J_0(\kappa r)\kappa\,\mathrm{d}\kappa - \frac{\cos(2\phi)}{4\pi r}\int_{\kappa=0}^{\infty}\frac{\zeta_s\tilde{g}^{te}_{zz;s}}{\bar{\Gamma}_s}J_1(\kappa r)\,\mathrm{d}\kappa\ , \tag{3}$$

$$\mathrm{TM} = \frac{\cos(2\phi)+1}{8\pi}\int_{\kappa=0}^{\infty}\frac{\Gamma_s\tilde{g}^{tm}_{hh;s}}{\eta_s}J_0(\kappa r)\kappa\,\mathrm{d}\kappa - \frac{\cos(2\phi)}{4\pi r}\int_{\kappa=0}^{\infty}\frac{\Gamma_s\tilde{g}^{tm}_{hh;s}}{\eta_s}J_1(\kappa r)\,\mathrm{d}\kappa\ . \tag{4}$$

Equations (108) and (109) in Hun15 yield the required parameters $\tilde{g}^{tm}_{hh;s}$ and $\tilde{g}^{te}_{zz;s}$,

$$\tilde{g}^{tm}_{hh;s} = P^{u-}_s W^u_s + P^{d-}_s W^d_s \ , \tag{5}$$

$$\tilde{g}^{te}_{zz;s} = \bar{P}^{u+}_s \bar{W}^u_s + \bar{P}^{d+}_s \bar{W}^d_s \ . \tag{6}$$

The parameters $P^{u\pm}_s$ and $P^{d\pm}_s$ are given in equations (81) and (82), $\bar{P}^{u\pm}_s$ and $\bar{P}^{d\pm}_s$ in equations (A-8) and (A-9); $W^u_s$ and $W^d_s$ in equation (74). This yields

$$\begin{aligned}
\tilde{g}^{te}_{zz;s} = & \frac{\bar{R}^+_s}{\bar{M}_s}\left\{\exp[-\bar{\Gamma}_s(z_s - z + d^+)] + \bar{R}^-_s \exp[-\bar{\Gamma}_s(z_s - z + d_s + d^-)]\right\} \\
& + \frac{\bar{R}^-_s}{\bar{M}_s}\left\{\exp[-\bar{\Gamma}_s(z - z_{s-1} + d^-)] + \bar{R}^+_s \exp[-\bar{\Gamma}_s(z - z_{s-1} + d_s + d^+)]\right\} \ , \\
= & \frac{\bar{R}^+_s}{\bar{M}_s}\left\{\exp[-\bar{\Gamma}_s(2z_s - z - z')] + \bar{R}^-_s \exp[-\bar{\Gamma}_s(z' - z + 2d_s)]\right\} \\
& + \frac{\bar{R}^-_s}{\bar{M}_s}\left\{\exp[-\bar{\Gamma}_s(z + z' - 2z_{s-1})] + \bar{R}^+_s \exp[-\bar{\Gamma}_s(z - z' + 2d_s)]\right\} \ ,
\end{aligned} \tag{7}$$

where $d^\pm$ is taken from the text below equation (67). There are four terms in the right-hand side, two in the first line and two in the second line. The first term in the first line is the integrand of $\mathrm{TE}^{+-}$, the second term in the first line corresponds to $\mathrm{TE}^{++}$, the first term in the second line is $\mathrm{TE}^{-+}$, and the second term in the second line is $\mathrm{TE}^{--}$.

If we look at $\mathrm{TE}^{+-}$, we have

$$\tilde{g}^{te+-}_{zz;s} = \frac{\bar{R}^+_s}{\bar{M}_s}\exp[-\bar{\Gamma}_s(2z_s - z - z')] \ , \tag{8}$$

and therefore

$$\begin{aligned}
\mathrm{TE}^{+-} = & \frac{\cos(2\phi) - 1}{8\pi}\int_{\kappa=0}^{\infty}\frac{\zeta_s \bar{R}^+_s}{\bar{\Gamma}_s \bar{M}_s}\exp[-\bar{\Gamma}_s(2z_s - z - z')]J_0(\kappa r)\kappa\,\mathrm{d}\kappa \\
& - \frac{\cos(2\phi)}{4\pi r}\int_{\kappa=0}^{\infty}\frac{\zeta_s \bar{R}^+_s}{\bar{\Gamma}_s \bar{M}_s}\exp[-\bar{\Gamma}_s(2z_s - z - z')]J_1(\kappa r)\,\mathrm{d}\kappa \ .
\end{aligned} \tag{9}$$

We can compare this to equation (4.165) in Book, with $\hat{I}^e_x = 1$ and slightly re-arranging it to look more alike, we get

$$\begin{aligned}
\hat{E}^{+-}_{xx;H} = & \frac{y^2}{4\pi r^2}\int_{\kappa=0}^{\infty}\frac{\zeta_1}{\Gamma_1}\frac{R^-_{H;1}}{M_{H;1}}\exp(-\Gamma_1 h^{+-})J_0(\kappa r)\kappa\mathrm{d}\kappa \\
& + \frac{x^2 - y^2}{4\pi r^3}\int_{\kappa=0}^{\infty}\frac{\zeta_1}{\Gamma_1}\left(\frac{R^-_{H;1}}{M_{H;1}} - \frac{R^-_{H;1}(\kappa=0)}{M_{H;1}(\kappa=0)}\right)\exp(-\Gamma_1 h^{+-})J_1(\kappa r)\mathrm{d}\kappa \\
& - \frac{\zeta_1(x^2 - y^2)}{4\pi\gamma_1 r^4}\frac{R^-_{H;1}(\kappa=0)}{M_{H;1}(\kappa=0)}\exp(-\gamma_1 R^{+-}) \ .
\end{aligned} \tag{10}$$

The equation is marked in blue to make it clear that the symbols and parameters in Hun15 and in Book are not exactly the same.

The difference between equations 9 and 10 is that the first one contains non-physical contributions. These have opposite signs in $\mathrm{TM}^{+-}$ and $\mathrm{TE}^{+-}$, and therefore cancel each other out. But if we want to know the specific contributions from TM and TE we have to remove them. The non-physical contributions only affect the $J_1$-integrals, and only for $\kappa = 0$.

The following lists for all 8 cases the term that has to be removed, in the notation of BOOK (for the notation as in HUN15 see the implementation in `tmtemod.py`):

$$TE^{++} = +\frac{\zeta_1(x^2-y^2)}{4\pi\gamma_1 r^4}\frac{\exp(-\gamma_1|h^-|)}{M_{H;1}(\kappa=0)} \ , \tag{11}$$

$$TE^{-+} = -\frac{\zeta_1(x^2-y^2)}{4\pi\gamma_1 r^4}\frac{R^+_{H;1}(\kappa=0)\exp(-\gamma_1 h^{-+})}{M_{H;1}(\kappa=0)} \ , \tag{12}$$

$$TE^{+-} = -\frac{\zeta_1(x^2-y^2)}{4\pi\gamma_1 r^4}\frac{R^-_{H;1}(\kappa=0)\exp(-\gamma_1 h^{+-})}{M_{H;1}(\kappa=0)} \ , \tag{13}$$

$$TE^{--} = +\frac{\zeta_1(x^2-y^2)}{4\pi\gamma_1 r^4}\frac{R^+_{H;1}(\kappa=0)R^-_{H;1}(\kappa=0)\exp(-\gamma_1 h^{--})}{M_{H;1}(\kappa=0)} \ , \tag{14}$$

$$TM^{++} = -\frac{\zeta_1(x^2-y^2)}{4\pi\gamma_1 r^4}\frac{\exp(-\gamma_1|h^-|)}{M_{V;1}(\kappa=0)} \ , \tag{15}$$

$$TM^{-+} = -\frac{\zeta_1(x^2-y^2)}{4\pi\gamma_1 r^4}\frac{R^+_{V;1}(\kappa=0)\exp(-\gamma_1 h^{-+})}{M_{V;1}(\kappa=0)} \ , \tag{16}$$

$$TM^{+-} = -\frac{\zeta_1(x^2-y^2)}{4\pi\gamma_1 r^4}\frac{R^-_{V;1}(\kappa=0)\exp(-\gamma_1 h^{+-})}{M_{V;1}(\kappa=0)} \ , \tag{17}$$

$$TM^{--} = -\frac{\zeta_1(x^2-y^2)}{4\pi\gamma_1 r^4}\frac{R^+_{V;1}(\kappa=0)R^-_{V;1}(\kappa=0)\exp(-\gamma_1 h^{--})}{M_{V;1}(\kappa=0)} \ . \tag{18}$$

Note that in equations 11 and 14 the correction terms have opposite sign as those in equations 15 and 18 because at $\kappa = 0$ the TM and TE mode correction terms are equal. Also note that in equations 12 and 13 the correction terms have the same sign as those in equations 16 and 17 because at $\kappa = 0$ the TM and TE mode reflection responses in those terms are equal but with opposite sign: $R^{\pm}_{V;1}(\kappa = 0) = -R^{\pm}_{V;1}(\kappa = 0)$.

HUN15 uses $\phi$, whereas BOOK uses $x$, $y$, for which we can use

$$\cos(2\phi) = -\frac{x^2-y^2}{r^2} \ . \tag{19}$$

## References


Hunziker, J., J. Thorbecke, and E. Slob, 2015, The electromagnetic response in a layered vertical transverse isotropic medium: A new look at an old problem: Geophysics, **80**, F1–F18, doi: 10.1190/geo2013-0411.1.

Ziolkowski, A. and E. Slob, 2017, CSEM book (TODO, add citation once the book is out): Cambridge University Press. ISBN: ??????.